Fachhochschule Graubünden
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Prior knowledge of mathematics: Test problems

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8.6.2023

For the mathematics modules in the technical Bachelor's degree programmes (Civil Engineering, Computational and Data Science, Mobile Robotics, Photonics) of the University of Applied Sciences of the Grisons, prior knowledge is required as follows:

Terms
- know the basic arithmetic operations, and be able to perform them.
- be able to rearrange and simplify terms.
- be able to expand and factorise terms.
- know the quadratic binomial theorems, and be able to apply them.
- be able to reduce and expand fractions.
- be able to rearrange and simplify terms with fractions.
- know the power rules, and be able to apply them to powers with integer or rational exponents.
- be able to determine domains of terms.

Sets
- know and understand what a set is.
- know and understand the basic concepts of set theory.
- know the basic operations of set theory, and be able to apply them.

Equations
- be able to solve linear equations in one variable.
- be able to solve linear inequations in one variable.
- be able to solve simple quadratic equations in one variable without quadratic formula.
- know the quadratic formula for solving quadratic equations in one variable.
- be able to solve quadratic equations in one variable using the quadratic formula.
- be able to solve simple goniometric equations in one variable.
- be able to solve simple exponential equations in one variable.
- be able to solve simple systems of equations.

Geometry/Trigonometry
- know the Pythagoras' theorem, and be able to apply it in simple problems.
- know the basic trigonometric relations in a right-angled triangle, and be able to apply them in simple problems.

Functions
- be able to calculate values of a basic function if the equation of the function is given.
- know and understand the graph as a representation of a function.
- be able to determine values and special points if the graph of a basic function is given.
- know and understand what a linear function is.
- know and understand what a quadratic function is.
- know and understand what the domain and the range of a function are.
- know and understand what a power function with a natural, integer, or rational exponent is.
- know and understand what an exponential function is.
- know and understand the basic trigonometric functions.
- know and understand the graphs of all mentioned types of functions.
- be able to treat simple problems regarding graphs of the mentioned types of functions.
- know and understand the relation between shifting and scaling of function graphs and the

corresponding operations to the function equations.
- be able to solve simple problems regarding shifting and scaling of function graphs.
- know and understand what the composition of functions is.
- be able to solve simple problems regarding the composition of functions.

Vectors
- know and understand what a vector is.
- know the basic operations of vector algebra, and be able to perform them both without coordinates and in Cartesian coordinates.
- be able to apply the basic operations of vector algebra in simple problems.

Concretely, it is expected to be able to solve the test problems below without aids (calculator, formulary, etc.):

Test problems

Terms

1. Calculate the expressions below:
   a) $2 + 3 \cdot 4$
   b) 3^{-2}
   c) -2^4
   d) $\sqrt{16}$

2. Simplify the expressions below:
   a) $7x - 5z + 10y + 3y + 8z - 4x$
   b) $(32m + 13q) - (14m + 7q)$
   c) $(15a - 2b) - (7a - (2a + b))$
   d) $5a^2b \cdot 4ab \cdot 3a^2b$
   e) $(x^3 - x^2y + xy^2 - y^3)(x + y)$

3. Expand the expressions below:
   a) $(p + q)^2$
   b) $(2x + 3y)^2$
   c) $(x - y)^2$
   d) $(2a - 3ax)^2$
   e) $(a + 2)(a - 2)$
   f) $(5xy + 3xz)(5xy - 3xz)$

4. Factorise the expressions below:
   a) $5a^2 - 10a^3 - 25a^4$
   b) $3a(x - a)^2 + 12a^2(x - a)$

5. Simplify the fractions below by reducing:

a) $\frac{14a}{18ab}$

b) $\frac{ab}{a^2b^2c}$

c) $\frac{8ab}{4a^2 - 4ab}$

d) $\frac{p^2 + p}{p^2 - 1}$

e) $\frac{x - y}{y - x}$

6. Rearrange the fractions below such that the denominator becomes $10a^2b^2x$:

a) $\frac{4y}{2a^2x}$

b) $\frac{5}{2ax}$

7. Rewrite the expressions below in one fraction:

a) $\frac{9x}{5} - \frac{6x}{5}$

b) $\frac{7x - 3y}{a} - \frac{2x + 5y}{a}$

c) $\frac{x}{2} + \frac{x}{3}$

d) $\frac{a}{b} - \frac{c}{ab}$

e) $\frac{a}{a - b} - \frac{b}{a^2 - b^2}$

f) $\frac{t + 7}{3t - 6} - \frac{t + 4}{t^2 - 2t}$

8. Simplify the expressions below:

a) $6 \cdot \frac{5}{12}$

b) $\frac{3}{4a} \cdot \frac{2}{9b}$

c) $\frac{d - 1}{18d} \cdot \frac{12d^2}{1 - d}$

d) $\frac{12pqr}{2pr}$

e) $\frac{16ab + 12aq}{4a}$

f) $\frac{30a^4b^3c^2}{5a^2bc}$

g) $\frac{-2x^2 - 4x}{-2x}$

h) $\frac{\frac{ax}{c}}{a}$

i) $\dfrac{\frac{a}{b^2}}{\frac{a^2}{b}}$

j) $\dfrac{x}{\frac{1}{y}}$

k) $\dfrac{r^2 + \frac{1}{r}}{r + \frac{1}{r^2}}$

9. Simplify the expressions below and write the answers without fractions:

a) $\dfrac{\left(a^2b^3a^4\right)^5}{\left(b^2a^3b^5\right)^2}$

b) $\left(\dfrac{a^{-1}b^2}{a^{-3}b^4}\right)^{-5}$

c) $\left(\dfrac{a^{1/2}\,b}{a^{-1/3}\,b^{11/8}}\right)^{-24/5}$

10. Determine all real numbers such that the expressions below are not defined:

a) $x^2 - 7$

b) $\dfrac{1}{x+2}$

c) $\sqrt{x+3}$

d) $\dfrac{1}{\sqrt{x^2-4}}$

Sets

11. Consider the sets below:

A = {1, 3, 5, 7}
B = {2, 4, 6, 8}
C = {2, 3, 4, 5}

Determine the sets below:

a) $B \cup C$

b) $B \cap C$

c) $(A \cup B) \setminus C$

12. Consider the sets below:

A = set of all cities of the world
B = set of all European cities
C = set of all coastal cities of the world

Judge whether the statements below are true or false:

a) Paris $\in$ B

b) Sydney $\notin$ C

c) $B \subset A$

d) $A \cap B = B$

e) $B \cap C = \{\}$

Equations

13. Solve the linear equations below for x:

a) $22(x - 11) - 5(x - 40) = 110 - (x + 53)$

b) $\frac{45}{2x - 9} - 2 = -\frac{27}{9 - 2x}$

c) $\frac{x}{x - 1} - \frac{x - 1}{x - 2} = 0$

d) $2a + cx = c - x$
Take into account that the parameters a and c can be any real numbers.

e) $|x + 3| = |x - 2| + 5$

14. Solve the linear inequations below for x:

a) $2(x - 1) - 5(x - 4) < 10 - (x + 5)$

b) $\frac{1}{x - 2} < \frac{3}{x + 1}$

15. Solve the quadratic equations below for x:

a) $2x^2 + 2x = 0$ (without quadratic formula)

b) $4(x - 2)^2 - 36 = 0$ (without quadratic formula)

c) $2x^2 - 7x + 3 = 0$

d) $x^2 - 6x + 9 = 0$

e) $5x^2 - 8x + 4 = 0$

f) $\frac{x - 4}{x - 5} = \frac{30 - x^2}{x^2 - 5x}$

16. Solve the goniometric equations below for x:

a) $\sin(x) = \frac{1}{2}$

b) $\cos\left(2x - \frac{\pi}{6}\right) = -\frac{1}{2}$

c) $\cos^2(x) = 1 - \sin(x)$

17. Solve the exponential equations below for x:

a) $10^x = 1000$

b) $3^x = 81$

c) $2^x = \frac{1}{16}$

d) $e^{3x} = 4$

e) $e^{2x} + e^x = 2$

18. Solve the systems of equations below for x and y:

a) $4x + 3y = 14$ (I)
$2x - y = 12$ (II)

b) $x + 2y = 3$ (I)
$4x + y^2 = 5$ (II)

Geometry/Trigonometry

19. In a triangle ABC, the sides and their lengths are denoted by a, b, and c. The side a is opposite to the corner A, b is opposite to B, and c is opposite to C. The heights and their lengths are denoted by h_a, h_b, and h_c.

a) The triangle is right-angled. a and b are the legs, c is the hypotenuse.
a and c are known.

Determine, without trigonometry, the length of side b …

i) … algebraically, i.e. without numerical values for a and c.

ii) … numerically (numerical values: $a = 5$, $c = 13$).

b) The triangle is not right-angled.
a, c, and h_c are known.

Determine, without trigonometry, the length of the side b …

i) … algebraically, i.e. without numerical values for a, c, and h_c.

ii) … numerically (numerical values: $a = 5$, $c = 4$, $h_c = 4$).

20. In a right-angled triangle ABC, the sides and their lengths are denotet by a, b, and c. The leg a is opposite to the corner A, leg b is opposite to B, and the hypotenuse c is opposite to C.

The internal angles and their values are denoted by α, β, and γ. α is at the corner A, β is at B, and γ ($\gamma = 90°$) is at C.

The heights and their lengths are denoted by h_a, h_b, and h_c.

a and h_c are known (numerical values: $a = 2$, $h_c = 1$).

Determine, with trigonometry, the lengths of the sides b and c, as well as the values of the internal angels α and β.

Functions

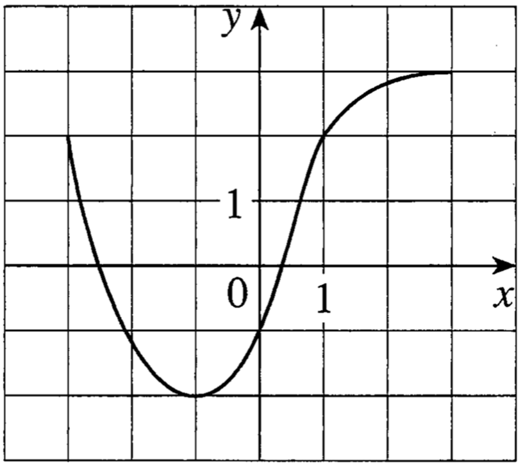
21. The equation of a function f is given as follows:

$$y = f(x) = 3x - 4$$

a) Determine both f(0) and f(-4).

b) Determine all values of x such that $f(x) = 0$.

22. The function f is defined on the interval $-3 \leq x \leq 3$. The graph of f is given as follows:



a) Determine f(-1).

b) Estimate the value of f(2).

c) Determine the values of x such that f(x) = 2.

d) Estimate the values of x such that f(x) = 0.

23. The graph of a linear function f contains the two points $P_1(-2|5)$ and $P_2(2|-4)$.

a) Determine the equation of the function f.

b) Determine the point where the graph of f and the y-axis intersect.

c) Determine the point where the graph of f and the x-axis intersect.

24. The graph of a quadratic function f contains …

a) … the three points $P_1(1|-1)$, $P_2(2|4)$, and $P_3(4|8)$.
Determine the general form of the equation of f.

b) … the two points $P_1(1|-8)$ and $P_2(2|-7)$, where P_1 is the vertex of the graph.
Determine the vertex form of the equation of f.

25. The graph of a quadratic function f has the vertex V(1|2) and touches the straight line y = 2x – 2 in exactly one point P.

Determine the coordinates of point P.

26. Judge whether the statements below are true or false:

a) The graph of the function $f: \mathbb{R} \to \mathbb{R}, x \mapsto y = f(x) = x^n$ ($n \in \mathbb{N}$) is axisymmetric with respect to the y-axis, if n is an even number.

b) The graph of the function $f: \mathbb{R} \to \mathbb{R}, x \mapsto y = f(x) = x^n$ ($n \in \mathbb{N}$) is point-symmetric with respect to the origin, if n is an odd number.

c) For the function $f: D \to \mathbb{R}, x \mapsto y = f(x) = x^{1/3}$, the number -8 is not an element of the domain D.

d) For the function $f: D \to \mathbb{R}, x \mapsto y = f(x) = (x + 1)^{-3/4}$, the number 0 is not an element of the domain D.

e) The graph of the function f: $\mathbb{R} \to \mathbb{R}$, $x \mapsto y = f(x) = a^x$ ($a \in \mathbb{R}^+\setminus\{1\}$) is completely above the x-axis.

f) For the function f: $\mathbb{R} \to \mathbb{R}$, $x \mapsto y = f(x) = e^{2x}$, the number 0 is not an element of the range.

g) The graph of the function f: $\mathbb{R} \to \mathbb{R}$, $x \mapsto y = f(x) = \sin(x)$ intersects the x-axis exactly at the positions $x_k = k\cdot 2\pi$ ($k\in\mathbb{Z}$).

h) The graph of the function f: $\mathbb{R} \to \mathbb{R}$, $x \mapsto y = f(x) = \cos(x)$ is, compared to the graph of the function g: $\mathbb{R} \to \mathbb{R}$, $x \mapsto y = g(x) = \sin(x)$, shifted by $\pi/2$ in the positive x-direction.

i) For the function f: $D \to \mathbb{R}$, $x \mapsto y = f(x) = \tan(x)$, the number $\pi/2$ is not an element of the domain D.

j) The graph of the function f: $D \to \mathbb{R}$, $x \mapsto y = f(x) = \tan(x)$ is point-symmetric with respect to the origin.

27. Consider the function f: $\mathbb{R} \to \mathbb{R}$, $x \mapsto y = f(x) = 2\sin(4x)$

Deterimine the equation of the function g, whose graph, compared to the graph of f, is …

a) … shifted by 2 units in the positive x-direction.

b) … shifted by 2 units in the negative x-direction.

c) … shifted by 2 units in the positive y-direction.

d) … shifted by 2 units in the negative y-direction.

e) … stretched by the factor 2 in the x-direction with respect to the origin.

f) … shrunk by the factor 2 in the x-direction with respect to the origin.

g) … stretched by the factor 2 in the y-direction with respect to the origin.

h) … shrunk by the factor 2 in the y-direction with respect to the origin.

28. Consider the function f: $\mathbb{R} \to \mathbb{R}$, $x \mapsto y = f(x) = \cos(2x)$

Determine by how many units the graph of the function g, compared to the graph of f, is shifted in the x-direction.

a) g: $\mathbb{R} \to \mathbb{R}$, $x \mapsto y = g(x) = \cos(2x - 6)$

b) g: $\mathbb{R} \to \mathbb{R}$, $x \mapsto y = g(x) = \cos(2x + 4)$

29. Consider the functions f and g below:

f: $\mathbb{R} \to \mathbb{R}$, $x \mapsto y = f(x) = x^2$
g: $\mathbb{R} \to \mathbb{R}$, $x \mapsto y = g(x) = 2x - 1$

Determine the expressions below:

a) $f(f(x))$

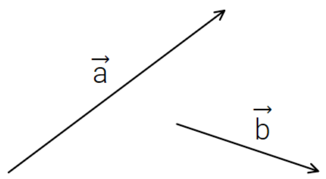
b) $g(g(x))$

c) $f(g(x))$

d) $g(f(x))$

Vectors

30. The two arrows below represent the two vectors $\vec{a}$ and $\vec{b}$:



Construct the arrows which represent the vectors below:

a) $\vec{a} + \vec{b}$

b) $\vec{a} - \vec{b}$

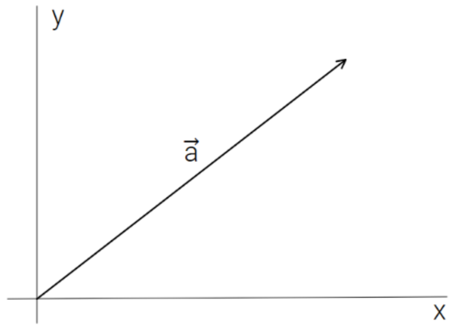
c) $\vec{b} - \vec{a}$

d) $2\vec{a}$

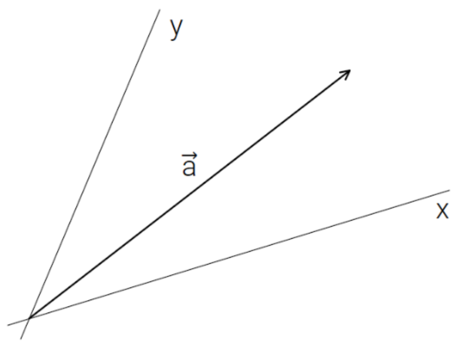
e) $-3\vec{b}$

31. Decompose the vector $\vec{a}$ into two components $\vec{a}_x$ and $\vec{a}_y$, i.e. $\vec{a} = \vec{a}_x + \vec{a}_y$. The component $\vec{a}_x$ should point in the x-axis, and the component $\vec{a}_y$ should point in the y-axis.

a)



b)



32. Consider the two vectors $\vec{a}$ and $\vec{b}$ below:

$\vec{a} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ $\vec{b} = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}$

Evaluate the expressions below:

a) $\vec{a} + \vec{b}$

b) $\vec{a} - \vec{b}$

c) $2\vec{a}$

d) $-3\vec{b}$

e) $-3\vec{a} + 4\vec{b}$

f) $|\vec{a}|$

g) $|\vec{a} - 2\vec{b}|$

33. Consider the three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ below:

$\vec{a} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ $\vec{b} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ $\vec{c} = \begin{pmatrix} 5 \\ -7 \\ 4 \end{pmatrix}$

The vector $\vec{c}$ should be expressed as a so-called linear combination of the two vectors $\vec{a}$ und $\vec{b}$, i.e.:

$\vec{c} = r \cdot \vec{a} + s \cdot \vec{b}$

Determine the two coefficients r and s.