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Prior knowledge of mathematics: Long answers

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Long answers to the test problems

Terms

1. a) $2 + 3 \cdot 4$

$= 2 + (3 \cdot 4)$

$= 2 + 12$

$= 14$

b) 3^{-2}

$= \frac{1}{3^2}$

$= \frac{1}{9}$

c) -2^4

$= -(2^4)$

$= -16$

d) $\sqrt{16}$

$= 4$ (only 4, not ± 4)

2. a) $7x - 5z + 10y + 3y + 8z - 4x$ — arranging addends according to x, y, and z

$= 7x - 4x + 10y + 3y - 5z + 8z$

$= (7x - 4x) + (10y + 3y) + (-5z + 8z)$

$= 3x + 13y + 3z$

b) $(32m + 13q) - (14m + 7q)$ — dissolving brackets

$= 32m + 13q - 14m - 7q$

$= 32m - 14m + 13q - 7q$

$= (32m - 14m) + (13q - 7q)$

$= 18m + 6q$

c) $(15a - 2b) – (7a - (2a + b))$ dissolving bracket $(2a + b)$

$= 15a – 2b – (7a – 2a – b)$ dissolving bracket

$= 15a – 2b – 7a + 2a + b$ arranging addends according to a and b

$= 15a – 7a + 2a – 2b + b$

$= (15a – 7a + 2a) + (- 2b + b)$

$= 10a + (- b)$

$= 10a – b$

d) $5a^2b \cdot 4ab \cdot 3a^2b$ arranging factors

$= 5·4·3·a^2·a·a^2·b·b·b$

$= 60·a^5·b^3$

$= 60a^5b^3$

e) $(x^3 - x^2y + xy^2 - y^3)(x + y)$ multiplying out

$= x^3x + x^3y – x^2yx – x^2yy + xy^2x + xy^2y – y^3x – y^3y$

$= x^4 + x^3y – x^3y – x^2y^2 + x^2y^2 + xy^3 – xy^3 – y^4$

$= x^4 + 0 + 0 + 0 – y^4$

$= x^4 - y^4$

3. a) $(p + q)^2$ applying 1st binomial theorem

$= p^2 + 2pq + q^2$

b) $(2x + 3y)^2$ applying 1st binomial theorem

$= (2x)^2 + 2·2x·3y + (3y)^2$

$= 4x^2 + 12xy + 9y^2$

c) $(x - y)^2$ applying 2nd binomial theorem

$= x^2 - 2xy + y^2$

d) $(2a - 3ax)^2$ applying 2nd binomial theorem

$= (2a)^2 – 2·2a·3ax + (3ax)^2$

$= 4a^2 - 12a^2x + 9a^2x^2$

e) $(a + 2)(a - 2)$ — applying 3rd binomial theorem

$= a^2 - 4$

f) $(5xy + 3xz)(5xy - 3xz)$ — applying 3rd binomial theorem

$= (5xy)^2 - (3xz)^2 = 25x^2y^2 - 9x^2z^2$

4. a) $5a^2 - 10a^3 - 25a^4$ — factorising common factor $5a^2$

$= 5a^2(1 - 2a - 5a^2)$

b) $3a(x - a)^2 + 12a^2(x - a)$ — factorising common factor $3a(x - a)$

$= 3a(x - a)((x - a) + 4a)$

$= 3(x - a)(x - a + 4a)$

$= 3a(x - a)(x + 3a)$

5. a) $\frac{14a}{18ab}$ — factorising both numerator and denominator

$= \frac{2 \cdot 7 \cdot a}{2 \cdot 9 \cdot ab}$ — reducing fraction by factors 2 and a

$= \frac{7}{9b}$

b) $\frac{ab}{a^2b^2c}$ — reducing fraction by factors a and b

$= \frac{1}{abc}$

c) $\frac{8ab}{4a^2 - 4ab}$ — factorising both numerator and denominator

$= \frac{2 \cdot 4 \cdot ab}{4a(a - b)}$ — reducing fraction by factors 4 and a

$= \frac{2b}{a - b}$

d) $\frac{p^2 + p}{p^2 - 1}$ — factorising both numerator and denominator

applying 3rd binomial theorem in denominator

$= \frac{p(p + 1)}{(p + 1)(p - 1)}$ — reducing fraction by factor p + 1

$= \frac{p}{p - 1}$

e) $\frac{x-y}{y-x}$ — factorising factor -1 in denominator

$= \frac{x-y}{-(x-y)}$ — reducing fraction by factor x – y

$= \frac{1}{-1}$

$= -1$

6. a) $\frac{4y}{2a^2x}$ — expanding fraction by factor $5b^2$

$= \frac{4y \cdot 5b^2}{2a^2x \cdot 5b^2}$

$= \frac{20b^2y}{10a^2b^2x}$

b) $\frac{5}{2ax}$ — expanding fraction by factor $5ab^2$

$= \frac{5 \cdot 5ab^2}{2ax \cdot 5ab^2}$

$= \frac{25ab^2}{10a^2b^2x}$

7. a) $\frac{9x}{5} - \frac{6x}{5}$

$= \frac{9x - 6x}{5}$

$= \frac{3x}{5}$

b) $\frac{7x - 3y}{a} - \frac{2x + 5y}{a}$

$= \frac{(7x - 3y) - (2x + 5y)}{a}$

$= \frac{7x - 3y - 2x - 5y}{a}$

$= \frac{5x - 8y}{a}$

c) $\frac{x}{2} + \frac{x}{3}$ — reducing fractions to a common denominator:
least common denominator = 6
expanding first fraction by 3
expanding second fraction by 2

$= \frac{x \cdot 3}{2 \cdot 3} + \frac{x \cdot 2}{3 \cdot 2}$

$= \frac{3x}{6} + \frac{2x}{6}$

$= \frac{3x + 2x}{6}$

$= \frac{5x}{6}$

d) $\frac{a}{b} - \frac{c}{ab}$

reducing fractions to a common denominator:
least common denominator = ab

expanding first fraction by a

$= \frac{a \cdot a}{b \cdot a} - \frac{c}{ab}$

$= \frac{a^2}{ab} - \frac{c}{ab}$

$= \frac{a^2 - c}{ab}$

e) $\frac{a}{a - b} - \frac{b}{a^2 - b^2}$

reducing fractions to a common denominator:
factorising denominators

applying 3rd binomial theorem in second denominator

$= \frac{a}{a - b} - \frac{b}{(a + b)(a - b)}$

reducing fractions to a common denominator:
least common denominator
$= (a + b)(a - b) = a^2 - b^2$

expanding first fraction by a + b

$= \frac{a(a + b)}{(a - b)(a + b)} - \frac{b}{(a + b)(a - b)}$

$= \frac{a(a + b) - b}{(a + b)(a - b)}$

$= \frac{a^2 + ab - b}{a^2 - b^2}$

f) $\frac{t + 7}{3t - 6} - \frac{t + 4}{t^2 - 2t}$

reducing fractions to a common denominator:
factorising denominators

$= \frac{t + 7}{3(t - 2)} - \frac{t + 4}{t(t - 2)}$

reducing fractions to a common denominator:
least common denominator = 3t(t - 2)

expanding first fraction by t

expanding second fraction by 3

$= \frac{(t + 7) \cdot t}{3(t - 2) \cdot t} - \frac{(t + 4) \cdot 3}{t(t - 2) \cdot 3}$

$= \frac{t^2 + 7t}{3t(t - 2)} - \frac{3t + 12}{3t(t - 2)}$

$= \frac{(t^2 + 7t) - (3t + 12)}{3t(t - 2)}$

$= \frac{t^2 + 7t - 3t - 12)}{3t(t - 2)}$

$= \frac{t^2 + 4t - 12)}{3t(t - 2)}$ factorising numerator

$= \frac{(t + 6)(t - 2)}{3t(t - 2)}$ reducing fraction by t – 2

$= \frac{t + 6}{3t}$

8. a) $6 \cdot \frac{5}{12}$

$= \frac{6 \cdot 5}{12}$

$= \frac{6 \cdot 5}{6 \cdot 2}$

$= \frac{5}{2}$

b) $\frac{3}{4a} \cdot \frac{2}{9b}$

$= \frac{3 \cdot 2}{4a \cdot 9b}$

$= \frac{3 \cdot 2}{2 \cdot 2 \cdot a \cdot 3 \cdot 3 \cdot b}$

$= \frac{1}{2 \cdot a \cdot 3 \cdot b}$

$= \frac{1}{6ab}$

c) $\frac{d - 1}{18d} \cdot \frac{12d^2}{1 - d}$

$= \frac{(d - 1) \cdot 12d^2}{18d(1 - d)}$

$= \frac{2 \cdot 6 \cdot (d - 1)d^2}{- 3 \cdot 6 \cdot d(d - 1)}$

$= \frac{2d}{- 3}$

$= - \frac{2d}{3}$

d) $\frac{12pqr}{2pr}$

$= \frac{2 \cdot 6 \cdot pqr}{2pr}$

$= \frac{6q}{1}$

$= 6q$

e) $\frac{16ab + 12aq}{4a}$

$= \frac{4a(4b + 3q)}{4a}$

$= \frac{4b + 3q}{1}$

$= 4b + 3q$

f) $\frac{30a^4b^3c^2}{5a^2bc}$

$= \frac{5 \cdot 6 \cdot a^4b^3c^2}{5a^2bc}$

$= \frac{6a^2b^2c}{1}$

$= 6a^2b^2c$

g) $\frac{-2x^2 - 4x}{-2x}$

$= \frac{-2x(x + 2)}{-2x}$

$= \frac{x + 2}{1}$

$= x + 2$

h) $\frac{\frac{ax}{c}}{a}$

$= \frac{\frac{ax}{c}}{\frac{a}{1}}$

$= \frac{ax}{c} \cdot \frac{1}{a}$

$= \frac{ax}{ac}$

$= \frac{x}{c}$

i) $\frac{\frac{a}{b^2}}{\frac{a^2}{b}}$

$= \frac{a}{b^2} \cdot \frac{b}{a^2}$

$= \frac{ab}{a^2b^2}$

$= \frac{1}{ab}$

j) $\dfrac{x}{\frac{1}{y}}$

$= \dfrac{\frac{x}{1}}{\frac{1}{y}}$

$= \dfrac{x}{1} \cdot \dfrac{y}{1}$

$= \dfrac{xy}{1}$

$= xy$

k) $\dfrac{r^2 + \frac{1}{r}}{r + \frac{1}{r^2}}$

$= \dfrac{\frac{r^2}{1} + \frac{1}{r}}{\frac{r}{1} + \frac{1}{r^2}}$

$= \dfrac{\frac{r^2 \cdot r}{1 \cdot r} + \frac{1}{r}}{\frac{r \cdot r^2}{1 \cdot r^2} + \frac{1}{r^2}}$

$= \dfrac{\frac{r^3}{r} + \frac{1}{r}}{\frac{r^3}{r^2} + \frac{1}{r^2}}$

$= \dfrac{\frac{r^3 + 1}{r}}{\frac{r^3 + 1}{r^2}}$

$= \dfrac{r^3 + 1}{r} \cdot \dfrac{r^2}{r^3 + 1}$

$= \dfrac{(r^3 + 1)r^2}{r(r^3 + 1)}$

$= r$

9. a) $\dfrac{\left(a^2 b^3 a^4\right)^5}{\left(b^2 a^3 b^5\right)^2}$

$= \dfrac{\left(a^2\right)^5 \left(b^3\right)^5 \left(a^4\right)^5}{\left(b^2\right)^2 \left(a^3\right)^2 \left(b^5\right)^2}$

$= \dfrac{a^{2 \cdot 5} b^{3 \cdot 5} a^{4 \cdot 5}}{b^{2 \cdot 2} a^{3 \cdot 2} b^{5 \cdot 2}}$

$= \dfrac{a^{10} b^{15} a^{20}}{b^4 a^6 b^{10}}$

$= \dfrac{a^{30} b^{15}}{a^6 b^{14}}$

$= a^{24} b$

b) $\left(\frac{a^{-1}b^2}{a^{-3}b^4}\right)^{-5}$

$= \left(\frac{a^3b^2}{ab^4}\right)^{-5}$

$= \left(\frac{a^2}{b^2}\right)^{-5}$

$= \frac{(a^2)^{-5}}{(b^2)^{-5}}$

$= \frac{a^{2\cdot(-5)}}{b^{2\cdot(-5)}}$

$= \frac{a^{-10}}{b^{-10}}$

$= a^{-10}b^{10}$

10. a) x^2 is defined for all real numbrers x. Furthermore, any real number can be subtracted from any real number. Therefore, the expression is defined for all real numbers x.

b) Division by zero is not defined. Hence, the denominator must not be zero. Therefore, the expression is not defined for $x = -2$.

c) The square root of a negative real number is not defined. Hence, the term under the root must not be smaller than zero. Therefore, the expression is not defined for $x < -3$.

d) Division by zero is not defined. Furthermore, the square root of a negative real number is not defined. Hence, the term under the root must be greater than zero. It follows that x^2 must be greater than 4, which means that x must be either greater than 2 or smaller than -2. Therefore, the expression is not defined for $-2 \leq x \leq 2$.

Equations

11. a)

$22(x - 11) - 5(x - 40) = 110 - (x + 53)$	multiplying out
$22x - 242 - (5x - 200) = 110 - (x + 53)$	dissolving brackets
$22x - 242 - 5x + 200 = 110 - x - 53$	simplifying on both sides
$17x - 42 = 57 - x$	+ x, + 42
$18x = 99$	: 18
$x = \frac{99}{18}$	
$= \frac{9\cdot 11}{9\cdot 2}$	
$= \frac{11}{2}$	

b)

$2a + cx = c - x$	+ x, - 2a
$cx + x = c - 2a$	factorising on left-hand side
$x(c + 1) = c - 2a$	: (c + 1)

$x = \frac{c - 2a}{c + 1}$

c)	$\frac{45}{2x - 9} - 2 = -\frac{27}{9 - 2x}$	factorising denominators
	$\frac{45}{2x - 9} - 2 = -\frac{27}{-(2x - 9)}$	
	$\frac{45}{2x - 9} - 2 = \frac{27}{2x - 9}$	$\cdot (2x - 9)$ (= least common multiple of all denominators)
	$\frac{45(2x - 9)}{2x - 9} - 2(2x - 9) = \frac{27(2x - 9)}{2x - 9}$	reducing on both sides
	$45 - 2(2x - 9) = 27$	multiplying out
	$45 - (4x - 18) = 27$	dissolving brackets
	$45 - 4x + 18 = 27$	simplifying on left-hand side
	$63 - 4x = 27$	$+ 4x, - 27$
	$36 = 4x$	interchanging sides
	$4x = 36$	$: 4$
	$x = 9$	

d)	$\frac{x}{x - 1} - \frac{x - 1}{x - 2} = 0$	$\cdot (x - 1)(x - 2)$ (= least common multiple of all denominators)
	$\frac{x(x - 1)(x - 2)}{x - 1} - \frac{(x - 1)(x - 1)(x - 2)}{x - 2} = 0(x - 1)(x - 2)$	reducing fractions
	$x(x - 2) - (x - 1)^2 = 0$	multiplying out
	$x^2 - 2x - (x^2 - 2x + 1) = 0$	dissolving brackets
	$x^2 - 2x - x^2 + 2x - 1 = 0$	simplifying
	$-1 = 0$	

This statement is false for any real number. Therefore, the equation has no solution.

Functions

12. a)

$f(0)$

$= 3 \cdot 0 - 4$

$= 0 - 4$

$= -4$

$f(-4)$

$= 3 \cdot (-4) - 4$

$= -12 - 4$

$= -16$

b) $0 = f(x) = 3x - 4$

$3x - 4 = 0 \qquad | + 4$

$3x = 4 \qquad | : 3$

$x = \frac{4}{3}$

13. a) x = -1, point P(-1|-2) on graph, hence y = f(-1) = -2

b) x = 2, point P(2|≈2.8) on graph, hence y = f(2) ≈ 2.8

c) y = f(x) = 2, points $P_1(-3|2)$ and $P_2(1|2)$ on graph, hence $x_1 = -3$ and $x_2 = 1$

d) y = f(x) = 0, points $P_1(\approx -2.5|0)$ and $P_2(\approx 0|0)$ on graph, hence $x_1 \approx -2.5$ and $x_2 \approx 0$

14. a) general equation of a linear function
$y = f(x) = ax + b$

$P_1(-2|5)$ and $P_2(2|-4)$ on graph (straight line with slope a)

slope $a = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4 - 5}{2 - (-2)} = \frac{-9}{4} = -\frac{9}{4}$

equation of function
$y = f(x) = y = f(x) = -\frac{9}{4}x + b$

$P_1(-2|5)$ on graph: $y = f(-2) = -\frac{9}{4} \cdot (-2) + b = 5$

equation for b
$\frac{9}{2} + b = 5$

solution
$b = \frac{1}{2}$

equation of function
$y = f(x) = -\frac{9}{4}x + \frac{1}{2}$

b) intersection point $S_y(0|y)$ on graph: $y = f(0) = -\frac{9}{4} \cdot 0 + \frac{1}{2} = \frac{1}{2}$
$S_y\left(0\middle|\frac{1}{2}\right)$

c) intersection point $S_x(x|0)$ on graph: $0 = f(x) = -\frac{9}{4}x + \frac{1}{2}$ with solution $x = \frac{2}{9}$
$S_x\left(\frac{2}{9}\middle|0\right)$